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THE DEGREES OF R.E. SETS WITHOUT THE UNIVERSAL SPLITTING PROPERTY

BY

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ABSTRACT. It is shown that every nonzero r.e. degree contains an r.e. set without the universal splitting property. That is, if δ is any r.e. nonzero degree, there exist r.e. sets $\varnothing <_T B <_T A$ with $\deg(A) = \delta$ such that if $A_0 \sqcup A_1$ is an r.e. splitting of A, then $A_0 \not\equiv_T B$. Some generalizations are discussed.

1. Introduction. A pair of r.e. sets A_0, A_1 are said to *split* an r.e. set A, written $A_0 \sqcup A_1 = A$, if $A_0 \cup A_1 = A$ and $A_0 \cap A_1 = \varnothing$. The types of degrees which may be realized as (halves of) splittings of r.e. sets have been analysed by many authors. For example, Sacks' splitting theorem **[Sa]** states that any nonrecursive r.e. set can be split into a pair of r.e. sets of incomparable Turing degree. Various extensions of this result are due to Owings **[Ow]**, Morley and Soare **[MS]** and Lachlan **[La]**. Also, there are the well-known results of Lachlan and of Ladner **[Ld1, Ld2]** on nonmitotic r.e. sets (sets which cannot be split into a pair of r.e. sets of the same Turing degree) and Ingrassia's **[In]** recent result that the degrees containing nonmitotic r.e. sets are dense.

Let A be an r.e. set. Let $S(A)$ and $N(A)$ be defined by

$$S(A) = \{\delta \mid \delta \text{ is an r.e. degree containing an r.e. set } A_0 \text{ such that there exists an r.e. set } A_1 \text{ with } A_0 \sqcup A_1 = A\},$$

and

$$N(A) = \{\delta \mid \delta \text{ is an r.e. degree and } \delta \notin S(A)\}.$$

We refer to $S(A)$ as the *splitting degrees of* A and $N(A)$ as the *nonsplitting degrees of* A. In **[LR1, LR2]**, Lerman and Remmel defined an r.e. set A to have the *universal splitting property* (USP), if $S(A) = \{\delta \mid \delta \leqslant \deg(A)\}$, that is, if every r.e. degree below $\deg(A)$ is a splitting degree of A. In **[LR1, LR2]**, Lerman and Remmel showed that r.e. sets may or may not have USP. Many of their results were extended in Ambos-Spies **[AS]**, Ambos-Spies and Fejer **[AF]**, Downey and Welch **[DW]** and Downey, Remmel and Welch **[DRW]**. For example, in **[DW]** and independently **[AS]**, an r.e. set A is constructed such that $N(A)$ is dense in the r.e. degrees (and also contains nontrivial intervals below $\deg(A)$). We shall say that an r.e. set (or degree)

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$B \leqslant_T A$ is a *nonsplitting witness* for A if $\deg(B) \notin S(A)$. Recently, Remmel and Shore (personal communication) have shown that if B is any nonzero incomplete r.e. set, then there is a complete r.e. set A such that B is a nonsplitting witness for A.

In this paper, we address ourselves to a fundamental question concerning non-USP sets:

Which r.e. degrees contain sets without USP?

Lerman and Remmel [**LR1**, **LR2**] have shown that the class of r.e. degrees containing r.e. sets without USP is a dense subset of the r.e. degrees and includes $\mathbf{0}'$. The results of Ambos-Spies [**AS**] and Downey and Welch [**DW**] show that this class also includes nontrivial initial segments of the r.e. degrees, and the results of Ambos-Spies show that if $\mathbf{a}$ is r.e. and low, then $\mathbf{a}$ is non-USP. In this paper we give the complete answer to this question by showing

THEOREM 2.1. *Every nonzero r.e. degree contains an r.e. set without USP.*

We also observe several generalizations of this result:

THEOREM 2.6. (i) *Let C, D be r.e. nonrecursive sets. Then there exist r.e. nonrecursive sets $A \equiv_T C$ and $B \leqslant_T D$ such that* $\deg(B) \notin S(A)$.

(ii) *Furthermore, in* (i) *we may also ensure that $A \leqslant_W C$ and $B \leqslant_W D$.*

(Here $\leqslant_W$ denotes weak truth table reducibility (cf. [**St**]).)

THEOREM 2.7. *Let δ be any nonzero r.e. degree. Then δ contains an r.e. set A such that $S(A)$ is not dense in* $[\mathbf{0}, \delta]$.

Moreover, in Theorem 2.8 we show that given any nonrecursive r.e. set D of degree δ, we can effectively compute r.e. sets $C <_T B <_T A$ with $A \equiv_T D$ and every degree in $[\deg(C), \deg(B)]$ a nonsplitting witness for A. Our results suggest several extensions and we analyse some of these. In view of the Remmel-Shore result quoted earlier, two quite natural extensions would be:

(1.1) Given any pair $\mathbf{a} <_T \mathbf{b}$ of r.e. nonzero degrees, there is an r.e. set B of degree $\mathbf{b}$ such that $\mathbf{a}$ is a nonsplitting witness for B.

(1.2) Given any nonzero r.e. degree $\mathbf{a}$, there is an r.e. set B of degree greater than $\mathbf{a}$ such that $\mathbf{a}$ is an *antisplitting witness* for B. Namely if $B_0 \sqcup B_1 = B$ is an r.e. splitting of B, then $\deg(B_0) \leqslant \mathbf{a}$ implies $B_0 \equiv_T \varnothing$ (cf. [**DW**]).

Unfortunately, as we shall later prove, both (1.1) and (1.2) fail. We had (in an earlier draft), proved weaker results by direct constructions, but here we shall establish these "plus splitting" results by using some results on W-degrees and some results from [**DW**]. As a further illustration of the surprising power of these "transfer" techniques, we shall give some further nonsplitting results, and some results about embedding nondistributive lattices in initial segments of the r.e. T-degrees.

Our notation is more or less standard and may be found in Soare [**So1**–**So3**]. Upper case italic letters stand for r.e. sets, and upper case Greek letters (for example Φ, Γ) for Turing reductions. "use$\{\cdots\}$" will mean the maximum element used in $\{\cdots\}$. For a set A and $z \in \omega$, $A[z]$ means $\{x \mid x \in A \text{ and } x \leqslant z\}$. Let $\langle\ ,\ \rangle$ be a standard pairing function of ω, and $\omega^{(e)} = \{\langle e, x\rangle : x \in \omega\}$. We identify sets with their characteristic functions (where necessary). Finally, one convention we shall use is that at any stage s of the constructions, all computations, use functions, followers, etc., will be bounded by s.

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2. Results.

THEOREM 2.1. *Every nonzero r.e. degree contains an r.e. set without the universal splitting property.*

PROOF. We shall build r.e. sets $A = \bigcup_s A_s$ and $B = \bigcup_s B_s$ in stages, so that B is a nonsplitting witness for A. Let C be a given r.e. set of nonzero degree, and let $g(\omega) = C$ be a 1-1 recursive enumeration of C. At each stage s, we let $\{a_{0,s} < a_{1,s} < \cdots\}$ list, in order, $\omega - A_s$. We ensure that $A \equiv_T C$ by permitting and coding (on the $a_{i,s}$). By this we mean we "code" C into A by promising to always have $a_{g(s),s} \in A_{s+1} - A_s$ at each stage s; and this will ensure that $C \leqslant_T A$. Similarly only allow elements to enter $A_{s+1} - A_s$ when C "permits" them to do so by asking that $z \in A_{s+1} - A_s$ only if $z \geqslant g(s)$, ensuring that $A \leqslant_T C$. We shall also ensure $B \leqslant_T A$ by permitting. For this construction, we will place x into $B_{s+1} - B_s$ only if $g(s) \leqslant x$.

We must meet the requirements

R_e: one of the following fails to hold

$$\begin{aligned}
&\text{(i)} \quad W_e \cap V_e = \varnothing,\\
&\text{(ii)} \quad W_e \cup V_e = A,\\
&\text{(iii)} \quad \Phi_e(W_e) = B,\\
&\text{(iv)} \quad \Gamma_e(B) = W_e,
\end{aligned}$$

where $\langle W_e, V_e, \Phi_e, \Gamma_e\rangle$ denotes an enumeration of quadruples of pairs of r.e. sets (W_e, V_e) and pairs of reductions (Φ_e, Γ_e).

For the sake of R_e we shall define a restraint $r(e, s)$ and the associated $R(e, s) = \max\{r(i, s) : i \leqslant s\}$. This will depend upon actions taken to meet R_e and the length of agreement $l(e, s)$ generated by (i)–(iv) above. Specifically,

$$l(e, s) = \max\{z\colon \text{for all } y \leqslant z, \text{(a), (b) and (c) below hold}\}:$$

$$\begin{aligned}
&\text{(a)} \quad (W_{e,s} \sqcup V_{e,s})[y] = A_s[y],\\
&\text{(b)} \quad \Phi_{e,s}(W_{e,s};\, y) = B_s(y),\\
&\text{(c)} \quad \Gamma_{e,s}(B_s;\, y) = W_{e,s}(y).
\end{aligned}$$

Our construction is possibly best motivated by analysing why the strategy used by Lerman and Remmel **[LR1]** to construct non-USP r.e. sets will not suffice here. In our terminology this construction consists of the following three steps (for one R_e):

Step 1. Pick a number x with a "trace" $T(x)$ with $x < T(x)$, $x \notin B_s$ and $T(x) \notin A_s$. Now declare x as a *follower* for R_e, and refrain from enumerating numbers $\leqslant T(x)$ into $A_t \cup B_t$ at stages $t > s$ until:

Step 2. The follower x becomes 0-*realized* (terminology from [**LR2**]). This means that a stage $s_1 > s$ occurs with $l(e, s_1) > T(x)$. Now we

(i) enumerate $T(x)$ into $A_{s_1+1} - A_{s_1}$,

(ii) raise the restraint $r(e, s_1 + 1)$ to s_1 (in view of our convention, this will exceed all numbers used in the computations (a)–(c) associated with $l(e, s)$),

(iii) assign $T'(x) = s_1 + 1$ to be a new trace for x.

The key observation is this: Suppose there is a stage $t > s_1$ such that

$$l(e, t) > r(e, s_1 + 1) = s_1,$$

and suppose further that our restraining was successful. Then if $W_e \sqcup V_e$ is truly a splitting of A, $T(x)$ must enter precisely one of $W_e \sqcup V_e$. Moreover $T(x)$ is the only such number $\leqslant s_1$ and furthermore we claim $T(x) \notin W_{e,t}$. Suppose $T(x) \in W_{e,t}$. Then by restraints we know from (c) in the definition of $l(e, s_1)$ that

$$\Gamma_{e,s_1}\big(B_{s_1}; T(x)\big) = \Gamma'_{e,t}\big(B_t; T(x)\big) = 0 \neq 1 = W_{e,t}(T(x)),$$

contrary to the fact that $l(e, t) > s_1 > T(x)$. Thus $T(x)$ must enter V_e. Thus we (temporarily) meet the R_e by:

Step 3. Wait for a stage $s_2 > s_1$ such that $l(e, s_2) > r(e, s_2 + 1) \geqslant s_1$. At this stage:

(i) enumerate $T'(x)$ into $A_{s_2+1} - A_{s_2}$,

(ii) maintain restraints,

(iii) enumerate x into $B_{s_2+1} - B_{s_2}$.

The point is that as $T(x)$ did not enter W_e, and since $T'(x) > s_1 = r(e, s)$, we know that by (b) (of $l(e, s)$)

$$\Phi_{e,s_1}\big(W_{e,s_1}; x\big) = \Phi_{e,s_2}\big(W_{e,s_2}; x\big) = \Phi_{e,s_2+1}\big(W_{e,s_2+1}; x\big) = 0 \neq 1 = B_{s_2+1}(x).$$

Furthermore, with priority e, restraints will preserve this disagreement forever. Following [**LR2**] we call stage s_2 a 1-*realization* stage.

By itself, permitting forces us to use many followers for one R_e. For example, it may be the case that by the stage s_1 when x is 0-realized, its trace $T(x)$ cannot be put into A_t for $t > s_1$ since $\forall t \geqslant s_1$ $(g(t) > T(x))$, and recall that we promised to allow z to enter $A_{t+1} - A_t$ only if $z \geqslant g(t)$. Furthermore, even if x gets 0-realized and acted on in Step 2, for the same reason it might be that once x gets 1-realized, $\forall t \geqslant s_2$ $(g(t) > T'(x))$. To overcome this we appoint an increasing set of followers and argue that if none eventually get 1-permitted, then C is recursive, contrary to hypothesis.

Our problems will stem from the *interaction* of permitting with the coding. Because we are coding, we will also promise to add $a_{g(s),s}$ to $A_{s+1} - A_s$. It may be that for each x which gets 1-realized and permitted, (say at stage t), it is also the case that at an even later stage $t' > t$,

$$a = a_{g(t'),t'} < \operatorname{use}\big\{\Phi_{e,s_2}\big(W_{e,s_2}; x\big)\big\}.$$

Since we have promised to add a to $A_{t'+1} - A_{t'}$, a might enter W_e and perhaps upset the "$\Phi_e(W_e; x) = 0 \neq 1 = B(x)$" computation from Step 3 which we were trying to preserve.

We remark that these problems can sometimes be overcome if we are permitting and coding with two sets C_1 and C_2 (rather than C), such that the permitting set C_2 is of strictly greater Turing degree than C_1. Thus in **[LR2]** matters are arranged so that eventually some 1-realized follower x gets permitted by C_2 after C_1 has stopped forcing small numbers (like "a" above) into A. This is arranged by "delayed permitting", and is one of the main ideas in the density theorem of **[LR2]** and similar density arguments (for example, [**In**, **Fe**]).

Our idea, roughly speaking, is to use a construction similar to the above, but to only perform a step similar to Step 3 at stage s when we can control the numbers which are not *already* in A_s. This control will be sufficient to ensure that *even if* a small number is coded in later, it would not matter because all of the numbers in a "critical region" are already in A_s.

Specifically, at each stage s we will not add just $a_{g(s),s}$ to $A_{s+1} - A_s$, but $a_{g(s),s}, \ldots, a_{g(s)+s,s}$. The idea then is this. Associated with R_e will be increasing set sequence of "positions" in $\omega - A_s$. The nth follower x, when appointed, will be assigned position $\langle e, n\rangle$. (Here we assume $\langle e, \ \rangle$ is monotone.) We shall choose $x \notin B_s$ as a follower of R_e if

(i) $x > \text{use}\{\Gamma_{e,s}(B_s; z) : z \leqslant a_{\langle e,n\rangle,s}\}$,

(ii) $l(e, s) > a_{x,s} \geqslant x$ and $x > \langle e, n\rangle$,

(iii) $l(e, s) > \text{use}\{\Phi_{e,s}(W_{e,s}; z) : z \leqslant x\}$.

(Figure 1, then, will be the picture with our intended "critical region".) At such a stage s, we *appoint* x as a follower and raise $r(e, s + 1) = s + 1$. Now we let $t(e, x) = a_{\langle e, n\rangle,s}$ and refer to $t(e, x)$ as the *confirmation target* of x. We shall do nothing else to satisfy R_e via x unless x gets *e-confirmed*. This means that at some stage $t > s$, $g(t) \leqslant \langle e, n\rangle$. By our convention, this will mean that $a_{g(t),t}, \ldots, a_{g(t)+t,t}$ all get enumerated into $A_{t+1} - A_t$. In particular, this means all numbers in the critical region get enumerated into A_{t+1}.

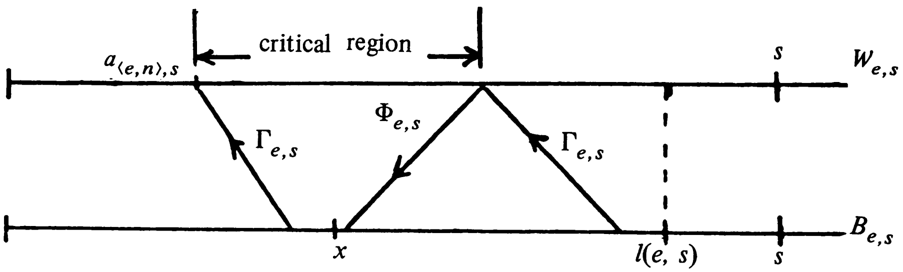

FIGURE 1

The key point is that after stage t, if $z \notin A_{t+1}$, then either $z \leqslant t(e, x)$ or $z > s$. In particular, if $z > s$, then $z > \text{use}\{\Phi_{e,s}(W_{e,q}; z); z \leqslant x\}$ for any stage $q > s$, provided that we keep maintaining our B-restraints originally imposed at stage s. Thus at the least stage $t' \geqslant t$ such that $l(e, t') \geqslant s$, we shall know that all of the computations we had at stage s have been maintained, and furthermore if z is a number in the critical region, then $z \in W_{e,t'}$ if $z \in W_e$. (In fact $z \in W_{e,s}$.) At this stage we declare x as *e-waiting*, and now know that if we ever get permitted to add x

to B_q for some stage $q > t'$, then we shall have satisfied R_e since only one of the two possibilities below can occur at every stage $t'' > t'$.

(i) No number $z \leqslant \text{use}\{\Phi_{e,s}(W_{e,s}; z): z \leqslant x\}$ enters $W_{e,t''}$, and so in particular, if we add x to B_q, then we know $\Phi_e(W_e; x) = 0 \neq 1 = B(x)$.

(ii) A number $y \leqslant \text{use}\{\Phi_{e,s}(W_{e,s}; x): z \leqslant x\}$ enters $W_{e,t''}$. Then in this case we know that $y \leqslant t(e, x)$. Since $x > \text{use}\{\Phi_{e,s}(B_{e,s}; z): z \leqslant m(e, x)\}$ we know that

$$\Gamma_{e,s}(B_{e,s}; y) = \Gamma_e(B; y) = 0 \neq 1 = W_e(y).$$

In either case we have a disagreement. In this way we can satisfy the R_e forever (with priority e).

We shall now give the details of the construction. We say that a requirement R_e is *satisfied at stage* s if either $W_{e,s} \cap V_{e,s} \neq \varnothing$, or one of the following hold for some $y \leqslant r(e, s)$:

(a) $\Gamma_{e,s}(B_s; y)\downarrow$ and $\Gamma_{e,s}(B_s; y) \neq W_{e,s}(y)$, and $\text{use}\{\Gamma_{e,s}(B_s; y)\} < r(e, s)$, or

(b) $\Phi_{e,s}(W_{e,s}; y)\downarrow$ and $\Phi_{e,s}(W_{e,s}; y) \neq B_s(y)$ and if $u = \text{use}\{\Phi_{e,s}(W_{e,s}; y)\}$, then $\Gamma_{e,s}(B_s; z) = W_{e,s}(z)$ for all $z \leqslant u$ and furthermore $\text{use}\{\Gamma_{e,s}(B_s; z)\} \leqslant r(e, s)$ for all $z \leqslant u$.

DEFINITION A. A requirement R_e *requires attention* at stage $s + 1$ if $e \leqslant s$, R_e is not satisfied at stage s and e is least such that one of the following hold.

(2.1) There is a follower x of R_e such that

(a) x is e-confirmed,

(b) $x > R(e - 1, s)$,

(c) x is e-waiting, and

(d) $g(s) \leqslant x$, or

(2.2) not (2.1) but there is a number $x \in \omega^{(e)}$ such that

(a) $l(e, s) > \max\{a_{x,s+1}, R(e, s)\}$,

(b) $x > \langle e, h(e, s)\rangle$,

(c) $l(e, s) > \text{use}\{\Phi_{e,s}(W_{e,s}; z): z \leqslant x\}$,

(d) $x > \text{use}\{\Gamma_{e,s}(B_s; z): z \leqslant a_{\langle e,h(e,s)\rangle,s+1}\}$.

CONSTRUCTION.

Stage 0. For all $y \in \omega$ set $h(y, 0) = 0$. Set $A_0 = B_0 = \varnothing$. Set $a_{e,0} = e$ for all $e \in \omega$.

Stage $s + 1$.

Step 1. Set $A_{s+1} = A_s \cup \{a_{g(s),s}, \dots, a_{g(s)+s,s}\}$, and set

$$a_{i,s+1} = \begin{cases} a_{i,s} & \text{for } i < g(s), \\ a_{i+s+1} & \text{otherwise.} \end{cases}$$

Step 2. For all $e \leqslant s$, and any follower x of R_e which is not already e-confirmed, if $a_{g(s),s} \leqslant t(e, x)$, then declare x as e-confirmed.

Step 3. For all $e \leqslant s$, and any follower x of R_e which is not already e-waiting, if (1) x is e-confirmed, and (2) $l(e, s) > R(e, s)$, then declare x as e-waiting.

Step 4 (i) If no R_e requires attention for all e, set $h(e, s + 1) = h(e, s)$ and set $R(e, s + 1) = R(e, s)$ and go to Stage $s + 2$.

(ii) If R_e requires attention with e least, there are 2 cases according to Definition A.

Case 1. (2.1) holds. Set $B_{s+1} = B_s \cup \{x\}$. Cancel all lower priority followers (that is of requirements R_j for $j > e$), and any followers of R_e. Set $r(e, s + 1) = s + 1$, $h(e, s + 1) = h(e, s)$.

Case 2. (2.2) holds. Appoint x as a follower of R_e. Cancel all lower priority followers. Set $h(e, s + 1) = h(e, s) + 1$, $t(e, x) = a_{\langle e,h(e,s)\rangle,s+1}$ and $r(e, s + 1) = s + 1$.

In either case, for all $j < e$ set $r(j, s + 1) = r(j, s)$ and for $f > e$ set $r(f, s + 1) = s + 1$. For all $k \neq e$, set $h(k, s + 1) = h(k, s)$. This is the end of the construction.

LEMMA 2.2. *$A \equiv_T C$, $B \leqslant_T C$ and* $\lim_s a_{i,s} = a_i$ *exists for all* i.

PROOF. (a) Given $z \in \omega$, the only way $z \in B_{s+1} - B_s$ for some s is by Step 4, Case 1 of the construction. This means that (2.1) holds, and in particular $g(s) \leqslant z$. Thus C-recursively find a stage $s(z)$ such that $\forall t > s(z)\ [g(t) > z]$. Then $z \in B$ iff $z \in B_{s(z)+1}$. Hence $B \leqslant_T C$.

(b) Now $a_{i,s+1} \neq a_{i,s}$ only if $g(s) \leqslant i$. As g is 1-1, this means $\lim_s a_{i,s}$ exists. Moreover, to determine if $z \in A$ or not, find a stage $t(z)$ such that $\forall s > t(z)$ $(g(s) > z)$. Then $\forall j \leqslant z\ (a_j = a_{j,t(z)})$. Moreover, by construction, $j \leqslant a_{j,s}$ for all j, s. Hence $z \in A$ iff $z \in A_{t(z)}$. Hence $A \leqslant_T C$.

(c) Finally $C \leqslant_T A$. A-recursively find a stage $n(z)$ such that $a_{z,n(z)} = a_{z+1}$. Then, by construction, we know that $\forall s > n(z)\ (g(s) > z)$. (For in Step 1 we enumerate $a_{g(s),s}$ into $A_{s+1} - A_s$.) Hence $z \in C$ iff $z \in C_{n(z)}$. □

To complete the verification, it remains to show that all the R_e require attention at most finitely often, are met and $\lim_s R(e, s) = R(e)$ exists.

Let e be least for which this statement fails to hold. Let t_0 be a stage such that $\forall s > t_0$ (R_j does not require attention at Stage s) and assume that $\forall s > t_0$ $(R(j, s) = R(j, t_0) = R(j))$ for all $j < e$. The proof will follow by the following sequence of lemmas.

LEMMA 2.3. *Suppose that s is a* (*least*) *stage with $s > t_0$ such that, for some follower x of R_e,* (2.1) *holds for x. Then*

(i) *R_e is met at Stage $s + 1$,*

(ii) $\forall t > s + 1$ (*R_e does not require attention at Stage t*),

(iii) $\forall t > s + 1\ (R(e, s + 1) = R(e, t) = s + 1)$.

PROOF. Let s, x, t_0 be as described above, and let x be the least such follower. As (2.1) pertains to x we know that x is e-confirmed, $x > R(e - 1, s)$, x is e-waiting and $g(s) \leqslant x$. Now at some stage $s_1 < s$, x was appointed by (2.2). Let $n = h(e, s_1)$. Then at Stage s_1 we know $t(e, x) = a_{\langle e,n\rangle,s_1}$. Furthermore, by construction, we also know

(i) $l(e, s_1 - 1) > \max\{a_{x,s}, x, R(e, s)\}$,

(ii) $x > \langle e, n\rangle$,

(iii) $l(e, s_1 - 1) > \text{use}\{\Phi_{e,s_1-1}(W_{e,s_1-1}; z): z \leqslant x\}$, and

(iv) $x > \text{use}\{\Gamma_{e,s_1-1}(B_{s_1-1}; z): z \leqslant t(e, x)\}$.

Now, at Stage s_1, as Case 2 applies, $R(e, s_1) = s_1$ (see Figure 1). Let

$$u = \max\left\{\text{use}\left\{\Gamma_{e,s_1-1}\left(B_{s_1-1}; z\right): z \leqslant p\right\}\right\},$$

where

$$p = \max\left\{\text{use}\left\{\Phi_{e,s_1-1}\left(W_{e,s_1-1}; z\right): z \leqslant t(e, x)\right\}\right\}.$$

Now, as $p \leqslant l(e, s_1 - 1)$, $u < s_1 - 1$. Now, at Stage s_1, as Case 2 applies, no number $\leqslant s_1$ is enumerated into $B_{s_1} - B_{s_1-1}$ and $R(e, s_1 - 1)$ is reset so that $R(e, s_1) = s_1 > u \geqslant p$.

Also, by (iv), if we set

$$M = \max\left\{\text{use}\left\{\Gamma_{e,s_1}\left(B_{s_1}; z\right): z \leqslant t(e, x)\right\}\right\},$$

then $x > M$ and $R(e, s_1) > s_1 > u \geqslant p \geqslant x > M$.

Now, as Case 2 applies, we know that if y is a follower of R_k and $s \geqslant t \geqslant s_1$ we have

(a) $k > e$ and so $y > s_1$ by the fact that in Step 4, lower priority followers are cancelled, or

(b) $k < e$ and so y was appointed before Stage t_0 and by choice of t_0, R_k cannot require attention at Stage t and in particular, $y \in B$ iff $y \in B_{t_0}$, or

(c) $k = e$ and so $y \notin B_s - B_{t_0}$ by choice of s as the least stage for which (2.1) holds.

This will mean that as x is e-waiting at Stage $s + 1$, $B_{s_1}[s_1] = B_{s_1-1}[s_1]$. Also, as x has been e-confirmed by Stage $s + 1$, for all y if $t(e, x) \leqslant y \leqslant s_1$, then as x is e-waiting (and so $l(e, s) > s_1$) we know $y \in W_e$ iff $y \in W_{e,s}$. Since the restraints have ensured that $B_s[u] = B_{s_1}[u]$, we know that $W_{e,s}[p] = W_{e,s_1}[p]$ and for all y with $t(e, x) \leqslant y \leqslant p, y \in W_e$ iff $y \in W_{e,s_1-1}$.

Consider Stage $s + 1$. As (2.1) holds, R_e will require attention via x (x least). We shall enumerate x into $B_{s+1} - B_s$ and set $R(e, s + 1) = s + 1$, and cancel all other followers of R_k for $k \geqslant e$. There are now two cases:

Case 1. $\forall t > s\ (W_{e,s}[p] = W_{e,t}[p])$. Then in this case there is a disagreement, via

$$0 = \Phi_{e,s}\left(W_{e,s}[p]; x\right) = \Phi_e(W_e; x) \neq 1 = B_{s+1}(x) = B(x).$$

Case 2. $\exists t > s\ (W_{e,s}[p] \neq W_{e,t}[p])$. Then in this case, there is a disagreement as follows: Let $y \leqslant p$ and suppose $y \in W_{e,t+1} - W_{e,t}$ for $t \geqslant s$. Then, by our analysis, $y < t(e, x)$. Now $x > M$ and, by restraints, $B_t[M] = B_{t+1}[M] = B_{s_1}[M]$. Hence

$$\begin{aligned}\Gamma_{e,s_1}\left(B_{s_1}[M]; y\right) &= \Gamma_{e,t+1}(B_{t+1}[M]; y) = \Gamma_e(B[M]; y)\\ &= \Gamma_e(B; y) = 0 \neq 1 = W_{e,t+1}(y) = W_e(y).\end{aligned}$$

Notice that one of Cases 1 or 2 must apply at Stage $s + 1$ and at all stages $t \geqslant s + 1$. This means R_e becomes satisfied at Stage $s + 1$ and remains so for all stages $t \geqslant s + 1$. Hence $\forall t > s\ (R(e, t) = R(e, s + 1) = s + 1)$ and R_e is met. □

We also note that if, at any stage $s > t_0$, we have $y \leqslant r(e, s)$ with R_e satisfied via y, then $\forall t > s\ [(r(e, t) = r(e, s))$ and R_e is met at stage $t]$ will hold. Since we suppose that R_e is not met, or requires attention infinitely often or $\lim_s r(e, s)$ does not exist, we shall suppose that R_e is *infinitely active* in the sense that at no stage $s > t_0$ does R_e become satisfied. This means that for any follower x of R_e, (2.1) does not pertain to x by Lemma 2.3. Now since we suppose that R_e fails, $l(e,s) \to \infty$. As $R(e,s)$ is reset only when (2.1) and (2.2) pertain, it must follow that if R_e is infinitely active, then R_e gets infinitely many followers appointed to it. These are never cancelled if they are appointed after Stage t_0. We claim

LEMMA 2.4. *R_e has infinitely many followers which are eventually e-confirmed.*

PROOF. Suppose not. We shall show that C is recursive contrary to hypothesis. We know that infinitely many followers get appointed to R_e. At any stage $s > t_0$, if we appoint a follower x to R_e such that $t(e,x) = a_{\langle e,n\rangle,s+1}$, then we reset $h(e,s+1) = n+1$. This process will continue for each new follower, and hence $h(e,s) \to \infty$ (e fixed). Now none of these followers get cancelled; thus if only finitely many are e-confirmed, there is a stage s_0 such that $s_0 \geqslant t_0$ and $\forall s > s_0$ (if x is a follower appointed at Stage s, then x is not e-confirmed).

Now this means $\forall s > s_0 \, \forall t > s\, [(x$ is appointed at stage s and $t(e,x) = a_{\langle e,m\rangle,s})$ $\to g(t) > \langle e,m\rangle]$. Given $z \in \omega$, to compute if $z \in C$, find a stage $s > s_0$ such that a follower $x(z)$ is appointed at Stage s with $t(e,x(z)) = a_{\langle e,m\rangle,s}$ and $\langle e,m\rangle > z$. Then $\forall t > s\,(g(t) > \langle e,m\rangle > z)$. Hence $z \in C \leftrightarrow Z \in C_s$. □

Let $F = \{y \mid y$ is a follower of R_e appointed at a stage $s > t_0$ such that y is eventually e-confirmed$\}$. Now as $l(e,s) \to \infty$, we know that for each $y \in F$, y eventually becomes e-waiting. The theorem will be proved once we establish

LEMMA 2.5. *Suppose $\forall y\,(y \in F \to$ (2.1) does not pertain to y). Then C is recursive.*

PROOF. Let $z \in \omega$. Find the least stage $s(z) > t_0$ and $y \in F$ such that:
(i) y is e-confirmed at Stage $s(z)$,
(ii) y is e-waiting at Stage $s(z)$,
(iii) $y > z$.
Then as (2.1) does not pertain, $\forall s > s(z)\,(g(s) > y > z)$. Thus $z \in C$ iff $z \in C_{s(z)}$ and so C is recursive. □

The theorem now follows because this means (2.1) pertains to some y and hence by Lemma 2.3 R_e will be met, $\lim_s R(e,s) = R(e)$ exists and R_e will stop requiring attention. □

REMARK. We also have the following corollary to the above (and some later results):

Let C be an r.e. nonrecursive set with $g(\omega) = C$, a 1-1 recursive enumeration of C. Define a set $A = \bigcup_s A_s$ in stages and at each stage s let $\overline{A_s} = \{a_{0,s} < a_{1,s} < \cdots\}$ via

$$A_{s+1} = A_s \cup \{a_{g(s),s}, \ldots, a_{g(s)+s,s}\}.$$

Set $A = \bigcup_s A_s$. Then A is non-USP.

We feel that this is an interesting phenomenon and perhaps deserves further investigation. A related result (perhaps) is the result of Ambos-Spies and Fejer [**AF**], that if B is an r.e. set which is a *cylinder*, then B has the property that if $C <_W B$ is r.e., then there is an r.e. splitting $B = B_0 \sqcup B_1$ with $B_0 \equiv_W C$. (Applying this to a "contiguous" r.e. T-degree gives an r.e. set B with USP.)

We shall now briefly discuss some generalizations of Theorem 2.1. Our first result comes from a slight modification of the proof above.

THEOREM 2.6. (i) *Let C, D be r.e. nonrecursive sets. Then there exists an r. e. set $A \equiv_T C$, and an r.e. set $B \leqslant_T D$ such that* $\deg(B) \notin S(A)$.

(ii) *In* (i) *we may ensure that also $A \leqslant_W C$ and $B \leqslant_W D$.*

PROOF. (i) Consider the proof of Theorem 2.1 with $g(\omega) = C$. Now, let f be a 1-1 recursive function with $f(\omega) = D$. We modify only (2.1) by replacing (d) by

(d′) $f(s) \leqslant x$.

The statement of Lemma 2.2 is changed to "$A \equiv_T C$, $B \leqslant_T D$ and $\lim_s a_{i,s} = a_i$ exists for all i". In part (a) of the proof of Lemma 2.2, we simply replace all occurrences of "g" by "f". Finally, in Lemma 2.5 we replace C by D and g by f, and the same proof will work.

(ii) This is because in Lemma 2.2, we prove $A \leqslant_T C$ and $B \leqslant_T D$ by simple permitting and this will give W-reduction. □

In particular then, if **a** and **b** are r.e. degrees with $0 <_T \mathbf{a} <_T \mathbf{b}$, there is an r.e. degree **c** and an r.e. set B of degree **b** such that $0 <_T \mathbf{c} \leqslant_T \mathbf{a}$, and **c** is a nonsplitting witness for B. So that means nonsplitting witnesses are "downward dense".

A natural question to ask is whether $S(A)$ is dense in $[\mathbf{0}, \deg(A)]$ for an r.e. set A?

The referee of [**DW**] observed the following:

PROPOSITION. *Let A be an r.e. non-USP set. Let B be an r.e. set with $B \leqslant_T A$ and B a nonsplitting witness for A. Then there exists an r.e. set $C <_T B$ such that for all r.e. sets D if $C \leqslant_T D \leqslant_T B$, then* $\deg(D) \notin S(A)$.

PROOF. Let A and B satisfy the hypotheses of the statement of the theorem. Sacks' split B as $B = B_0 \sqcup B_1$ with B_i r.e. and $B_0 |_T B_1$. Then either $C = B_0$ or $C = B_1$ will satisfy the theorem. For suppose not. Then we may r.e. split A as $A = A_0 \sqcup E_0 = A_1 \sqcup E_1$ with $B_i \leqslant_T A_i \leqslant_T B$. Now $B \equiv_T A_0 \cup A_1$ and $(A_0 \cup A_1) \cup (E_0 \cap E_1)$ is an r.e. splitting of A, contradicting the choice of B. □

Thus we have

THEOREM 2.7. *Let δ be any nonzero r.e. degree. Then δ contains an r.e. set A such that $S(A)$ is not dense in* $[\mathbf{0}, \delta]$.

We remark that the "nonsplitting interval" obtained from the proposition cannot be effectively found because of the Sacks' splitting. We can modify our original strategy to be able to compute A, B and C effectively. We sketch a proof of this:

THEOREM 2.8. *Let D be an r.e. nonrecursive set. Then we can effectively find r.e. sets C, B and A such that $A \equiv_T D$, $\varnothing <_T C <_T B <_T A$ and for all r.e. sets E if $C \leqslant_T E \leqslant_T B$, then E is a nonsplitting witness for A.*

PROOF. We modify the proof of Theorem 2.1. Let $g(\omega) = D$ instead of $g(\omega) = C$ this time. The requirements are

R_{2e}: $\Phi_e(C) \neq B$, and

R_{2e+1}: It is not the case that $W_e \sqcup V_e = A$,

$\Phi_e(W_e) = C$ and $\Gamma_e(B) = W_e$, whilst ensuring $C \leqslant_T B \leqslant_T A$. We meet the R_{2e} by a Friedberg-Muchnik procedure with permitting. Thus we have a follower x "targeted" for B and we wait for a stage s to occur with $\Phi_{e,s}(C_s; x)\downarrow$, and say x is realized, and set $r(2e, s) = s$. If $\Phi_{e,s}(C_s; x) \neq 0$, then R_{2e} is met (since $B_s(x) = 0$). If $\Phi_{e,s}(C_s; x) = 0$, we will act when x gets permitted. We say x gets permitted if $g(t) \leqslant x$ at some stage $t \geqslant s$. When permitted we enumerate x into B, maintain restraints and know that $\Phi_{e,t}(C_t; x) = 0 \neq B(x) = 1$. We keep appointing new followers until one gets realized and permitted, or one does not get realized (in which case $\Phi_e(C) \neq B$). We argue that if infinitely many get realized but none get permitted, then C is recursive in the same manner as Lemma 2.4.

For the R_{2e+1} we proceed almost as we did in Theorem 2.1, except that a follower x is added to both C and B, and we must impose our restraint on both C and B. Thus, we proceed as in Theorem 2.1 making the appropriate changes ("C" for "B" in some places, $2e + 1$ for e and modifying the definition of $l(e, s)$ to $l(2e + 1, s)$) and the details follow for the same reasons as they did in the original result. We ask the reader to supply the remaining details. □

As Jockusch observed, the techniques of Lerman and Remmel [**LR1**, **LR2**] extend to show that the r.e. degrees containing r.e. sets without the *universal weak truth table reduction property* (UWP) are dense in the r.e. degrees. We say an r.e. set A has UWP if, for all r.e. $B \leqslant_T A$, there exists an r.e. set $C \equiv_T B$ with $C \leqslant_W A$. However, our techniques do not extend to produce r.e. sets without UWP in each r.e. degree. Indeed, Ladner and Sasso [**LS**] have shown that below any given nonzero r.e. (Turing) degree, there is an r.e. *contiguous* degree, that is, one consisting of only one r.e. W-degree. Obviously, any r.e. set of contiguous degree has UWP. Also, any set of complete W-degree has UWP. It remains an open question whether or not these are the only types of sets with UWP. It is also unclear whether or not every r.e. USP set occurs in such degrees. We remark that each of these questions could be answered negatively by constructing the appropriate set of "incomplete but not low$_2$" degree, since Cohen (cf. Stob [**St**]) has shown that contiguous r.e. degrees are low$_2$. (There are low$_2$ but not low$_1$ contiguous degrees, cf. [**AF**].)

We remark that in [**DW**], Downey and Welch analysed r.e. sets with what is called the *antisplitting property*, meaning there is an r.e. set $\varnothing <_T B <_T A$ such that if $A_0 \sqcup A_1 = A$ and $A_0 \leqslant_T B$, then $A_0 \equiv_T \varnothing$. Downey and Welch could produce such sets in a subset of the cappable degrees (namely the degrees containing (strongly) atomic r.e. sets). Does every r.e. degree contain an r.e. set with the antisplitting property? Currently, this question is open (even) for cappable degrees.

Remmel and Shore (personal communication) have shown that if $\mathbf{a}$ is an r.e. degree if $\mathbf{0} <_T \mathbf{a} <_T \mathbf{0}'$, then there is an r.e. set B of degree $\mathbf{0}'$ with $\mathbf{a} \notin S(B)$. This together with Theorem 2.4 suggests the question as to whether Theorem 2.4 can be combined with the Remmel-Shore result. That is:

(*) Given r.e. degrees $\mathbf{0} <_T \mathbf{b} <_T \mathbf{a}$, does there exist an r.e. set A of degree $\mathbf{a}$ such that $\mathbf{b} \notin S(A)$?

We shall now show that (*) has a negative solution, in a rather strong way. We shall say that $\mathbf{a}$ *plus splits* $\mathbf{b}$ if $\mathbf{a} \leqslant \mathbf{b}$ and, given any r.e. set B of degree $\mathbf{b}$, there exists an r.e. splitting $B_0 \sqcup B_1$ such that $\deg(B_0) = \mathbf{a}$.

THEOREM 2.9. *Let C be an r.e. set of high degree. There exists an r.e. nonrecursive set $B \leqslant_T C$ such that if* $\mathbf{a} \in S(B)$, *then* $\mathbf{a}$ *plus splits* $\deg(B)$.

PROOF. Recall from **[DW]** (or **[AS]**) an r.e. set B is called strongly atomic (or antimitotic) if, whenever $B = B_0 \sqcup B_1$ is an r.e. splitting of B, the infinum of degrees of B_0 and of B_1 exists and is $\mathbf{0}$. In **[DW]** it is shown that if C is a high r.e. set, then there is an r.e. set $B \leqslant_T C$ such that B is strongly atomic, nonrecursive and of contiguous degree.

Now, let $\mathbf{a} \in S(B)$ and let D be any r.e. set with $D \equiv_T B$. By contiguity, $D \equiv_W B$. Now as $\mathbf{a} \in S(B)$, there exist r.e. sets A_0, A_1 such that $B = A_0 \sqcup A_1$ with $\deg(A_0) = \mathbf{a}$. Without loss of generality, we may suppose $\varnothing <_T A_0 <_T B$. Consequently, by strong atomicity $\deg(A_0)$, $\deg(A_1)$ form a minimal pair of r.e. degrees, with supremum $\deg(B)$. We now shall apply a result of Lachlan, called in **[DW]** *Lachlan's lemma*: namely, if E, F and G are r.e. sets with $E \leqslant_W F \oplus G$, then there exists an r.e. splitting $E = E_0 \sqcup E_1$ of E with $E_0 \leqslant_W F$ and $E_1 \leqslant_W G$.

Now $A_0 \sqcup A_1 = B \equiv_W D$. Hence, by Lachlan's lemma, $D = D_0 \sqcup D_1$ with $D_0 \leqslant_W A_0$ and $D_1 \leqslant_W A_1$. We claim $D_0 \equiv_W A_0$, giving the result (since $\deg(a_0) = \mathbf{a}$). To see this, as $D_1 \leqslant_W A_1$, it follows that $D_0 \oplus A_1 \equiv_W B$. Now, by contiguity, $A_0 \leqslant_W D_0 \oplus A_1$. Hence $A_0 = H_0 \sqcup H_1$ with $H_0 \leqslant_W D_0$ and $H_1 \leqslant_W A_1$. Thus, $H_1 \leqslant_W A_1, A_0$. But, by strong atomicity, $\deg(A_0)$, $\deg(A_1)$ form a minimal pair and so $H_1 \equiv_T \varnothing$. Hence $A_0 \equiv_T H_0 \leqslant_T D_0 \leqslant_T A_0$ giving the desired result. □

We remark that the above proof actually shows that:

(i) For all *W-degrees* $\mathbf{a} \in S(B)$ if $D \equiv_T B$ is r.e., then $D = D_0 \sqcup D_1$ with D_0 of W-degree $\mathbf{a}$.

(ii) If B is an r.e. contiguous, strongly atomic set, then $S(B) = \{\mathbf{d} \mid \mathbf{d} = \mathbf{0} \vee \mathbf{d} = \deg(B) \vee \exists \mathbf{e}\, (\mathbf{e} |_T \mathbf{d} \wedge (\mathbf{e} \cap \mathbf{d} = \mathbf{0}) \wedge (\mathbf{e} \oplus \mathbf{d} = \deg(B)))\}$.

In some sense, the use of contiguous r.e. strongly atomic sets to prove the above result is somewhat surprising, in view of the very strong *nonsplitting* properties possessed by such sets. For example, we quote the following

THEOREM 2.10. *Let A be an r.e. nonrecursive contiguous strongly atomic set. Then*:

(i) *$S(A)$ gives an embedding of the countable atomless boolean algebra into the contiguous r.e. T- and W-degrees, preserving sups and infs with greatest degree* $\deg(A)$ *and least* $\mathbf{0}$.

(ii) *$S(A)$ is* ***nowhere dense*** *in the r.e. degrees, that is, if* $\mathbf{c} <_T \mathbf{d}$, *then there exist* $\mathbf{c}'$, $\mathbf{d}'$ *with* $\mathbf{c} <_T \mathbf{c}' <_T \mathbf{d}' <_T \mathbf{d}$ *such that* $[\mathbf{c}, \mathbf{d}] \cap S(A) = \varnothing$.

(iii) *A has the* ***antisplitting property***; *that is, there exists an r.e. set B with $\varnothing <_T B <_T A$ such that if $A_0 \sqcup A_1 = A$ is an r.e. splitting of A, then $A_0 \leqslant_T B$ implies $A_0 \equiv_T \varnothing$. B is called an* ***antisplitting witness*** *for A.*

(iv) *In fact, if $\varnothing <_T B <_T A$ is an r.e. set, there exists an r.e. set B' with $\varnothing <_T B' \leqslant_T B$ such that B' is an antisplitting witness for A.*

(v) *Finally, there exists an r.e. set C with $C <_T A$ such that if $A_0 \sqcup A_1 = A$ is an r.e. splitting of A, then $C \leqslant_T A_0$ implies $A_0 \equiv_T A$.*

PROOF. (i) is [**DW**, Theorem 5.5], (ii) is [**DW**, Theorem 5.4], (iii) is [**DW**, Corollary 4.2] and (v) is an unpublished result of Downey, Ingrassia, Stob and Welch, whose proof will appear elsewhere. We prove (iv).

Let $\varnothing <_T B <_T A$. Let B' be a contiguous r.e. set with $\varnothing <_T B' <_W B$ such that B' is a W-anticupping witness for B; namely if Q is r.e. and $B' \oplus Q\ {}_W\!\geqslant B$, then $Q\ {}_W\!\geqslant B$. Such a B' exists by a result of Ladner and Sasso [**LS**]. Now suppose $A = A_0 \sqcup A_1$ is an r.e. splitting of A with $\varnothing <_T A_0 <_T B'$. By contiguity of B', $A_0 <_W B'$. It follows that $B' \oplus A_1 \equiv_W A$ since A is contiguous (and thus $B \leqslant_W A$). Thus $B' \oplus A_1\ {}_W\!\geqslant B$. As B' is a W-anticupping witness for B, $A_1\ {}_W\!\geqslant B$. Hence $A_1\ {}_W\!\geqslant B\ {}_W\!> B'\ {}_W\!> A_0\ {}_W\!> \varnothing$, contradicting the strong atomicity of A: A_0, A_1 are supposedly a minimal pair. □

The use of W-degrees and r.e. sets with special splitting properties may obtain several other results. We illustrate with two further examples.

In view of the Remmel-Shore result, another natural conjecture would be that if $\varnothing <_T \mathbf{a} <_T \mathbf{0}'$, then $\mathbf{a}$ is an antisplitting witness for some r.e. set. We call a degree $\mathbf{a}$ *persistent* if, for all r.e. sets B with $\mathbf{a} <_T \deg(B)$, there exists an r.e. splitting $B_0 \sqcup B_1 = B$ with $\mathbf{0} <_T \deg(B_0) \leqslant_T \mathbf{a}$.

THEOREM 2.11 (i) *There exists an r.e. low persistent degree* $\mathbf{a}$.

(ii) *Furthermore, we may construct low r.e. degrees* $\mathbf{a}_0$, $\mathbf{a}_1$ *such that for all r.e. sets* C, *if* $\deg(C) \nleqslant \mathbf{a}_0$ *and* $\deg(C) \nleqslant \mathbf{a}_1$, *then* $C = C_0 \sqcup C_1$ *with* C_0, C_1 *nonrecursive and* $\mathbf{0} <_T \deg(C_i) <_T \mathbf{a}_i$. *In particular*, $\mathbf{a}_0$ *is "persistent" for all non-low r.e. sets.*

PROOF. Let K be an r.e. set of complete W-degree. Sacks' split K as $K = K_0 \sqcup K_1$ with the K_i low. Let $\mathbf{a}_i = \deg(K_i)$. Let C be any r.e. set with $C \nleqslant K_i$ for $i = 1, 2$. As $C \leqslant_W K_0 \oplus K_1$, by Lachlan's lemma $C = C_0 \sqcup C_1$ with $C_i \leqslant_W K_i$. Now suppose, for example, that $C_0 \equiv_T \varnothing$. Then $C \equiv_W C_1 \leqslant_W K_1$, a contradiction, giving the result. □

As our final illustration, we shall give a result concerning initial segments of the r.e. T-degrees. We say an r.e. degree $\mathbf{a} \neq \mathbf{0}$ *bounds a* 1-3-1 *lattice* if there exist r.e. degrees $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2 \leqslant_T \mathbf{a}$, with $\mathbf{a}_i |_T \mathbf{a}_j$ and $\mathbf{a}_i \cap \mathbf{a}_j = \mathbf{0}$ for $i \neq j$, and $\mathbf{a}_k \leqslant_T \mathbf{a}_i \oplus \mathbf{a}_j$ for $i \neq j \neq k$. By a result of Lachlan, as there is an r.e. degree bounding no minimal pairs, there is an r.e. degree bounding no 1-3-1 lattice. However, using strongly atomic r.e. sets and W-degrees, we get the following somewhat surprising definability result.

THEOREM 2.12. *Below any high r.e. degree, there exists an r.e. degree* $\mathbf{a} \neq \mathbf{0}$ *such that*

(i) *every r.e. degree* $\mathbf{b}$ *with* $\mathbf{0} <_T \mathbf{b} \leqslant_T \mathbf{a}$ *is the sup of a minimal pair, and*

(ii) $\mathbf{a}$ *bounds no* 1-3-1 *lattice.*

PROOF. Let A be an r.e. strongly atomic contiguous degree of degree $\mathbf{a}$. It is shown in [**AS**] that if $\mathbf{b} \leqslant_T \mathbf{a}$, then $\mathbf{b}$ is strongly atomic. In fact in [**AS**] it is proved that:

($*$) If C and D are r.e. sets with $C \leqslant_W D$, then there exists an r.e. set $C' \equiv_W C$ such that if $C' = C_0 \sqcup C_1$, then there exist r.e. sets D_0, D_1 with $D = D_0 \sqcup D_1$ and $C_i \leqslant_W D_i$.

Now suppose A_0, A_1, A_2 are r.e. sets whose degrees form a 1-3-1 lattice below **a**. Let $E = A_0 \oplus A_1 \oplus A_2$. By (*) $E \equiv_W F$ with F r.e. and F strongly atomic (as, by contiguity, $E \leqslant_W A$).

Now as $F \equiv_W E$, by Lachlan's lemma, $F = F_0 \sqcup F_1 \sqcup F_2$ with $F_i \leqslant_W A_i$. We claim $F_i \equiv_W A_i$. Let $i = 0$. Now $A_0 \leqslant_W F_0 \oplus F_1 \oplus F_2$ and so $A_0 = H_0 \sqcup H_1 \sqcup H_2$ with $H_j \leqslant_W F_j \leqslant A_j$. In particular for $j \neq 0$, $H_j \leqslant_W A_j, A_0$. As deg(A_1), deg(A_2), deg(A_3) form a 1-3-1 lattice for $j = 1, 2$, $H_j \equiv_T \varnothing$. Thus $A_0 \equiv_W H_0 \leqslant_W F_0 \leqslant_W A_0$, and so $F_0 \equiv_W A_0$ as required. Similarly $A_i \equiv_W F_i$ for $i = 0, 1, 2$. But this is impossible since then $\varnothing <_T F_0 \leqslant_T F_1 \sqcup F_2$ and $F = F_0 \sqcup (F_1 \cup F_2)$ is an r.e. splitting of a supposedly strongly atomic r.e. set. □

A surprising number of quite strong definability results for the r.e. T-degrees, and splitting type results may be found by the use of similar techniques. Namely, we build r.e. sets with certain degree theoretic splitting properties and then we analyse structural interactions of the r.e. T- and W-degrees to force some properties to hold in, say, the r.e. T-degrees. It would seem an interesting project to analyse the extent to which such "transfer" techniques may be used. We refer the reaer to [**DS**] for some results along these lines.

ADDED IN PROOF. We have recently extended Theorem 2.10 to show that if A is an r.e. nonrecursive contiguous strongly atomic set, there exists an r.e. set B with $\varnothing <_T B <_T A$ such that if $A_0 \sqcup A_1 = A$ is an r.e. splitting of A, then $B \leqslant_T A_0$ implies $A_0 \equiv_T A$ and $A_0 \leqslant_T B$ implies $A_0 \equiv_T \varnothing$. The proof also uses W-degrees.

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